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September 30th, 2008

Dear Student,

Thank you for picking up one of the Manhattan GMAT Strategy Guides—we hope that it refreshes your memory of the junior-high math that you haven't used in years. Maybe it will even teach you a new thing or two.

As with most accomplishments, there were many people involved in the various iterations of the book that you're holding. First and foremost is Zeke Vanderhoek, the founder of Manhattan GMAT. Zeke was a lone tutor in New York when he started the Company in 2000. Now, eight years later, MGMAT has Instructors and offices nationwide, and the Company contributes to the studies and successes of thousands of students each year.

These 3rd Edition Strategy Guides have been refashioned and honed based upon the continuing experiences of our Instructors and our students. We owe much of these latest editions to the insight provided by our students. On the Company side, we are indebted to many of our Instructors, including but not limited to Josh Braslow, Dan Gonzalez, Mike Kim, Stacey Koprince, Jadran Lee, Ron Purewal, Tate Shafer, Emily Sledge, and of course Chris Ryan, the Company's Lead Instructor and Director of Curriculum Development.

At Manhattan GMAT, we continually aspire to provide the best Instructors and resources possible. We hope that you'll find our dedication manifest in this book. If you have any comments or questions, please e-mail me at andrew.yang@manhattangmat.com. I'll be sure that your comments reach Chris and the rest of the team—and I'll read them too.

Best of luck in preparing for the GMAT!

Sincerely,

Andrew Yang
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NUMBER PROPERTIES

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INTEGERS

Integers are "whole" numbers, such as 0, 1, 2, and 3, that have no fractional part. Integers can be positive (1, 2, 3…), negative (−1, −2, −3…), or the number 0.

The GMAT uses the term integer to mean a non-fraction or a non-decimal. The special properties of integers form the basis of most Number Properties problems on the GMAT.

Arithmetic Rules

Most arithmetic operations on integers will always result in an integer. For example:

$4 + 5 = 9$	$(-2) + 1 = -1$	The sum of two integers is always an integer.
$4 - 5 = -1$	$(-2) - (-3) = 1$	The difference of two integers is always an integer.
$4 \times 5 = 20$	$(-2) \times 3 = -6$	The product of two integers is always an integer.

Divisibility questions test on whether the result of division of integers results in an integer.

However, division is different. Sometimes the result is an integer, and sometimes it is not:

$8 \div 2 = 4$, but $2 \div 8 = \frac{1}{4}$

$(-8) \div 4 = -2$, but $(-8) \div (-6) = \frac{4}{3}$

The quotient of two integers is SOMETIMES an integer.

An integer is said to be **divisible** by another number if the integer can be divided by that number with an integer result (meaning that there is no remainder). For example, 21 is divisible by 3 because when it is divided by 3, an integer results ($21 \div 3 = 7$), but 21 is not divisible by 4 because when it is divided by 4, a non-integer results ($21 \div 4 = 5.25$).

Alternatively, we can say that 21 is divisible by 3 because 21 divided by 3 yields 7 with zero remainder. On the other hand, 21 is not divisible by 4 because 21 divided by 4 yields 5 with a remainder of 1.

Here are some more examples:

$8 \div 2 = 4$	Therefore, 8 is divisible by 2.
$2 \div 8 = 0.25$	Therefore, 2 is NOT divisible by 8.
$(-6) \div 2 = -3$	Therefore, −6 is divisible by 2.
$(-6) \div (-4) = 1.5$	Therefore, −6 is NOT divisible by −4.

Note that there is no remainder when 8 is divided by 2, or when −6 is divided by 2, but there IS a remainder when 2 is divided by 8, or −6 is divided by −4.

We can also say that 2 is a **divisor** or **factor** of 8.

Rules of Divisibility by Certain Integers

The Divisibility Rules are important shortcuts to determine whether an integer is divisible by 2, 3, 4, 5, 6, 8, 9, and 10. The GMAT frequently tests whether you have internalized these rules (especially on its more challenging questions), so it is important to memorize them and be able to call upon them at any time.

It is a good idea to memorize the rules for divisibility by 2, 3, 4, 5, 6, 8, 9 and 10.

An integer is divisible by:

2 if the integer is EVEN.
12 is divisible by 2, but 13 is not. Integers that are divisible by 2 are called "even" and integers that are not are called "odd." You can tell whether a number is even by checking to see whether the units (ones) digit is even. Thus, 1,234,567 is odd, because 7 is odd, whereas 2,345,678 is even, because 8 is even. Even and Odd number properties are discussed in depth in Chapter 2 of this book.

3 if the SUM of the integer's DIGITS is divisible by 3.
72 is divisible by 3 because the sum of its digits is 9, which is divisible by 3. By contrast, 83 is not divisible by 3, because the sum of its digits is 11, which is not divisible by 3.

4 if the integer is divisible by 2 TWICE, or if the LAST TWO digits are divisible by 4.
28 is divisible by 4 because it is divisible by 2 once (=14) and then again (=7). For larger numbers, check only the last two digits. For example, 123,456 is divisible by 4 because 56 is divisible by 4, whereas 345,678 is not divisible by 4 because 78 is not divisible by 4.

5 if the integer ends in 0 or 5.
75 and 80 are divisible by 5, but 77 and 83 are not.

6 if the integer is divisible by BOTH 2 and 3.
48 is divisible by 6 since it is divisible by 2 (it ends with an 8, which is even) AND by 3 ($4 + 8 = 12$, which is divisible by 3).

8 if the integer is divisible by 2 THREE TIMES, or if the LAST THREE digits are divisible by 8.
32 is divisible by 8 since it is divisible by 2 once (16), twice (8), and a third time (4). For larger numbers, check only the last 3 digits. For example, 123,456 is divisible by 8 because 456 is divisible by 8 ($456 \div 2 = 228$, $228 \div 2 = 114$, and $114 \div 2 = 57$), whereas 123,556 is not divisible by 8 because 556 is not divisible by 8 ($556 \div 2 = 278$ and $278 \div 2 = 139$, but 139 is not divisible by 2).

9 if the SUM of the integer's DIGITS is divisible by 9.
4,185 is divisible by 9 since the sum of its digits is 18, which is divisible by 9. By contrast, 3,459 is not divisible by 9, because the sum of its digits is 21, which is not divisible by 9.

10 if the integer ends in 0.
670 is divisible by 10, but 675 is not.

The GMAT can also test these divisibility rules in reverse. For example, if you are told that a number has a ones digit equal to 0, you can infer that that number is divisible by 10. Similarly, if you are told that the sum of the digits of x is equal to 21, you can infer that x is divisible by 3 but NOT divisible by 9.

Note also that there is no rule listed for divisibility by 7. A rule for divisibility by 7 exists, but it is cumbersome. The simplest way to check for divisibility by 7, or by any other number not found in this list, is to perform long division.

Factors and Multiples

Factors and Multiples are essentially opposite terms.

A factor is a positive integer that divides evenly into an integer, so 1, 2, 4 and 8 are all the factors (also called divisors) of 8.

A multiple of an integer is formed by multiplying that integer by any integer, so 8, 16, 24, and 32 are some of the multiples of 8. Additionally, negative multiples are possible (–8, –16, –24, –32, etc.), but the GMAT does not test negative multiples directly. Also, zero (0) is technically a multiple of every number, because zero divided by any number yields zero, which is an integer.

Note that an integer is always both a factor and a multiple of itself, and that 1 is a factor of **every** integer.

There are several different ways that the GMAT can state that x is divisible by y—learn these different phrasings and mentally convert them to divisibility when you see them!

Fewer Factors, More Multiples

Sometimes it is easy to confuse factors and multiples. The mnemonic "Fewer Factors, More Multiples" should help you remember the difference. Factors divide into an integer and are therefore less than or equal to that integer. Positive multiples, on the other hand, multiply out from an integer and are therefore greater than or equal to that integer.

Any integer only has a limited number of factors. For example, there are only four factors of 8: 1, 2, 4, and 8. By contrast, there is an infinite number of multiples of an integer. For example, the first 5 positive multiples of 8 are 8, 16, 24, 32, and 40, but you could go on listing multiples of 8 forever.

Factors, multiples, and divisibility are very closely related concepts. For example, 3 is a factor of 12. This is equivalent to saying that 12 is a multiple of 3. **If x is a factor of y, then y must be a multiple of x**. Similarly, 12 is divisible by 3. **If x is a factor of y, then y must be divisible by x.**

On the GMAT, this terminology is often used interchangeably—sometimes within the same problem—in order to make the problem seem harder than it actually is. The key is to be aware of the different ways that the GMAT can phrase information about divisibility. Moreover, you should try to convert all such statements to the same terminology. For example, **all** of the following statements **say exactly the same thing**:

- 12 is divisible by 3
- 12 is a multiple of 3
- $\frac{12}{3}$ is an integer
- 3 is a divisor of 12
- $12 = 3n$, where n is an integer
- 3 is a factor of 12
- $\frac{12}{3}$ yields a remainder of 0 **(remainders discussed later in this chapter)**

On the GMAT, a problem may say that "x is a divisor of 12," or "12 divided by x yields a remainder of 0." In both of these cases, you may want to rephrase any such statement to "12 is divisible by x." That way, you can think through the problem in the same way, regardless of the GMAT's phrasing of the problem.

Throughout the rest of this chapter, we will use different phrasings of this concept to get you used to the different ways in which divisibility information can be conveyed.

Learn the divisibility rules for addition and subtraction!

Divisibility and Addition/Subtraction

The GMAT can test you on divisibility across sums and differences. The three key rules to know are listed below. For each rule, assume that N is an integer.

(1) If you add or subtract two or more multiples of N, the result is a multiple of N.
$18 + 15 = 33$ (Multiple of 3) + (Multiple of 3) = (Multiple of 3)
(This should make sense: $6 \cdot 3 + 5 \cdot 3 = 11 \cdot 3$.)

(2) If you add a multiple of N to a non-multiple of N, the result is a non-multiple of N. (The same holds true for subtraction.)
$18 - 10 = 8$ (Multiple of 3) − (Non-multiple of 3) = (Non-multiple of 3)

(3) If you add two non-multiples of N, the result could either a multiple of N or a non-multiple of N.
$19 + 13 = 32$ (Non-multiple of 3) + (Non-multiple of 3) = (Non-multiple of 3)
$19 + 14 = 33$ (Non-multiple of 3) + (Non-multiple of 3) = (Multiple of 3)
The exception to this rule is when $N = 2$. This exception is covered in the "Odds & Evens" chapter of this book.

Consider the following examples:

Is $64 + 40$ divisible by 8?

First, observe that 64 is divisible by 8, and 40 is also divisible by 8. Since both numbers are divisible by 8, the sum must also divisible by 8. Furthermore, notice that the difference between 64 and 40 ($64 - 40 = 24$) is also divisible by 8.

Is N divisible by 7?

(1) $N = x - y$, where x and y are integers
(2) x is divisible by 7, and y is not divisible by 7.

Statement (1) tells us that N is the difference between two integers (x and y), but it does not tell us anything about the divisibility of x or y by 7. INSUFFICIENT.

Statement (2) tells us nothing about N. INSUFFICIENT.

Statements (1) and (2) combined tell us that x is a multiple of 7, but y is not a multiple of 7. The difference between x and y can NEVER be divisible by 7 if x is divisible by 7 but y is not. (If you are not convinced, try testing it out by picking numbers.) SUFFICIENT: N cannot be a multiple of 7.

The correct answer is (C): BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

Introduction to Primes

Prime numbers are a very important topic on the GMAT. A prime number is any positive integer larger than 1 with exactly two factors: 1 and itself. In other words, a prime number has NO factors other than 1 and itself. For example, 7 is prime because the only factors of 7 are 1 and 7. However, 8 is not prime because it is divisible by 2 and 4.

Note that the number 1 is not considered prime, as it has only one factor (itself). Thus, the first prime number is 2, which is also the only even prime. This special property of the number 2 is discussed further in the "Odds & Evens" chapter of this book. The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29. You should memorize these primes (or be able to reproduce them).

A prime number is an integer larger than 1 that has only two factors: 1 and itself.

Here are some additional facts about primes that may be helpful on the GMAT:

(1) **There is an infinite number of prime numbers.** There is no upper limit to the size of prime numbers. As you look at larger and larger numbers, primes become more and more rare. Far to the right on the number line, you can find very long strings of consecutive integers, none of which are prime. However, there are always primes greater than any specified value.

(2) **There is no simple pattern in the prime numbers.** Since 2 is the only even prime number, all other primes are odd. However, there is no easy pattern to determining which odd numbers will be prime. Each number needs to be tested directly to determine whether it is prime.

(3) **Positive integers with only two factors must be prime, and positive integers with more than two factors are never prime.** Any integer greater than or equal to 2 has at least two factors: 1 and itself. Thus, if there are only two factors of x (with x equal to an integer ≥ 2), then the factors of x MUST be 1 and x. Therefore, x must be prime. Also, do not forget that the number 1 is NOT prime. The number 1 has only one factor (itself), so it is defined as a non-prime number.

These facts can be used to disguise the topic of prime numbers on the GMAT.

What is the value of integer x?

(1) x has exactly 2 factors.
(2) When x is divided by 2, the remainder is 0.

Statement (1) indicates that x is prime, because it has only 2 factors. This statement is insufficient by itself, since there are infinitely many prime numbers. Statement (2) indicates that 2 divides evenly into x, meaning that x is even; that is also insufficient by itself. Taken together, however, the two statements reveal that x must be an even prime—and the only even prime number is 2. The answer is (C): BOTH statements TOGETHER are sufficient, but NEITHER statement ALONE is sufficient.

If x is a prime number, what is the value of x?

(1) There are a total of 50 prime numbers between 2 and x, inclusive.
(2) There is no integer n such that x is divisible by n and $1 < n < x$.

At first, this problem seems outlandishly difficult. How are we to list out the first 50 prime numbers in under 2 minutes?

Remember, however, that this is a **Data Sufficiency** problem. We do not need to list the first 50 primes. Instead, all we need to do is determine WHETHER we can do so.

For Statement (1), we know that certain numbers are prime and others are not. We also know that x is prime. Therefore, if we were to list all the primes from 2 on up, we eventually would find the 50th-largest prime number. That number must equal x, because x is prime, so it MUST be the 50th item on that list of primes. This information is SUFFICIENT.

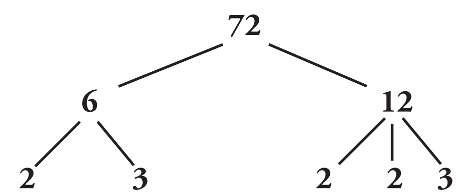
For Statement (2), we are told that x is not divisible by any integer greater than 1 but less than n. Therefore, x can only have 1 and x and factors. In other words, x is prime. We already know this result, in fact: it was given to us in the question stem. So Statement (2) does not help us determine what x is. INSUFFICIENT.

Every integer larger than 1 has a unique prime factorization.

The correct answer is (A): Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient. (Incidentally, for those who are curious, the 50th prime number is 229.)

Prime Factorization

One very helpful way to analyze a number is to break it down into its prime factors. This can be done by creating a prime factor tree, as shown to the right with the number 72. Simply test different numbers to see which ones "go into" 72 without leaving a remainder. Once you find such a number, then split 72 into factors. For example, 72 is divisible by 6, so it can be split into 6 and 72 ÷ 6, or 12. Then repeat this process on the factors of 72 until every branch on the tree ends at a prime number. Once we only have primes, we stop, because we cannot split prime numbers into two smaller factors. In this example, 72 splits into 5 total prime factors (including repeats): $2 \times 3 \times 2 \times 2 \times 3$.



Prime factorization is an extremely important tool to use on the GMAT. One reason is that once we know the prime factors of a number, we can determine ALL the factors of that number. The factors can be found by building all the possible products of the prime factors. This technique is described in detail later, in the Advanced section of this chapter.

An easier way to find all the factors of a small number is to use **factor pairs**. Factor pairs for any integer are the pairs of factors that, when multiplied together, yield that integer.

In order to determine the factor pairs of any given number N, you should generally start with the automatic factors: 1 and N. (1 and N are always factors of any integer N.) Then, simply "walk upwards" from 1, testing to see whether different numbers are factors of N. Once you find a number that is a factor of N, find its partner factor by dividing N by the factor. Keep walking upwards until all factors are exhausted.

<u>For example, the factors of 72 can be determined as follows:</u>

(1) Make a table with 2 columns labeled “Small” and “Large.”
(2) Start with 1 in the small column and 72 in the large column.
(3) Test the next possible factor of 72 (which is 2). 2 is a factor of 72, so write “2” underneath the “1” in your table. Divide 72 by 2 to find the factor pair: $72 \div 2 = 36$. Write “36” in the large column.
(4) Test the next possible factor of 72 (which is 3). Repeat this process until the numbers in the small and the large columns run into each other. In this case, once we have tested 8 and found that 9 was its factor pair, we can stop. (The small column will always end at or just before the square root of the number you are factoring. $\sqrt{72}$ is between 8 and 9, since $8^2 = 64$ and $9^2 = 81$. Thus, the small column ends at 8.)

Small	Large
1	72
2	36
3	24
4	18
6	12
8	9

Factor pairs can be used to determine ALL of the factors of any integer.

This method can be effective for a number such as 72, which is relatively small and has few factors. However, the process can be difficult for larger numbers, which may have many more factors. Other methods of counting factors are discussed later in this chapter.

On the GMAT, prime factorization is useful for many other applications in addition to enumerating factors. Some other situations in which you might need to use prime factorization include the following:

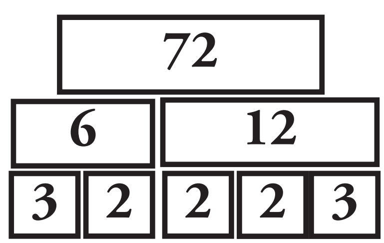
(1) Determining whether one number is divisible by another number
(2) Determining the greatest common factor of two numbers
(3) Reducing fractions
(4) Finding the least common multiple of two (or more) numbers
(5) Simplifying square roots
(6) Determining the exponent on one side of an equation with integer constraints

This list could go on and on. Prime numbers are the building blocks of integers. Many problems require variables to be integers, and you can often solve or simplify these problems by analyzing primes. A simple rule to remember is this: **if the problem states that a number is an integer, or if the problem implicitly requires the number to be an integer, you may need to use prime factorization to solve the problem**.

Factor Foundation Rule

The GMAT expects you to know the factor foundation rule: **if *a* is a factor of *b*, and *b* is a factor of *c*, then *a* is a factor of *c***. In other words, any integer is divisible by all of its factors—and it is also divisible by all of the FACTORS of its factors.

For example, if 72 is divisible by 12, then 72 is also divisible by all the factors of 12 (1, 2, 3, 4, 6, and 12). Written another way, if 12 is a factor of 72, then all the factors of 12 are also factors of 72. The Factor Foundation Rule allows us to conceive of factors as building blocks in a foundation. 12 and 6 are factors, or building blocks, of 72 (because 12×6 builds 72).



The number 12, in turn, is built from its own factors; for example, 4×3 builds 12. Thus, if 12 is part of the foundation of 72 and 12 in turn rests on the foundation built by its prime factors (2, 2, and 3), then 72 is also built on the foundation of 2, 2, and 3.

Going further, we can build almost any factor of 72 out of the bottom level of the foundation. For instance, we can see that 8 is a factor of 72, because we can build 8 out of the three 2's in the bottom row ($8 = 2 \times 2 \times 2$).

We say **almost** any factor, because one of the factors cannot be built out of the building blocks in the foundation: the number 1. Remember that the number 1 is not prime, but it is still a factor of every integer. Except for the number 1, every factor of 72 (or of any other integer) can be built out of the lowest level of the number's building blocks.

Think of the prime factors of an integer as that integer's "foundation," from which all factors of that number (except 1) can be built.

The Prime Box

The easiest way to work with the Factor Foundation Rule is with a tool called a Prime Box. A Prime Box is exactly what its name implies: a box that holds all the prime factors of a number (in other words, the lowest-level building blocks). Here are prime boxes for 72, 12, and 125:

72	**12**	**125**
2, 2, 2, 3, 3	2, 2, 3	5, 5, 5

Notice that we must repeat copies of the prime factors if the number has multiple copies of that prime factor.

You can use the prime box to test whether or not a specific number is a factor of another number.

Is 27 a factor of 72?

72
2, 2, 2, 3, 3

$27 = 3 \times 3 \times 3$. But we can see that 72 only has <u>two</u> 3's in its prime box. Therefore we cannot make 27 from the prime factors of 72. Thus, 27 is not a factor of 72.

Given that the integer n is divisible by 3, 7, and 11, what other numbers must be divisors of n?

n
3, 7, 11, ... ?

Since we know that 3, 7, and 11 are prime factors of n, we know that n must also be divisible by all the possible **products** of the primes in the box: 21, 33, 77, and 231.

Without even knowing what n is, we have found 4 more of its factors: 21, 33, 77, and 231.

You should use a COMPLETE prime box for the prime factorization of a *number*, but a PARTIAL prime box for the prime factorization of a *variable*.

Notice also the ellipses and question mark ("... ?") in the prime box of n. This reminds us that we have created a **partial prime box** of n. Whereas the COMPLETE set of prime factors of 72 can be calculated and put into its prime box, we only have a PARTIAL list of prime factors of n, because n is an unknown number. We know that n is divisible by 3, 7, and 11, but we do NOT know what additional primes, if any, n has in its prime box.

Most of the time, when building a prime box for a VARIABLE, we will use a partial prime box, but when building a prime box for a NUMBER, we will use a complete prime box.

Is p divisible by 168?

(1) p is divisible by 14
(2) p is divisible by 12

The first step in this kind of problem is to determine what prime factors p needs in order to be divisible by 168. The prime factorization of 168 is $2 \times 2 \times 2 \times 3 \times 7$, so the question can be restated as follows:

168
2, 2, 2, 3, 7

Are there at least three 2's, one 3, and one 7 in the prime box of *p*?

Statement (1): *p*
2, 7, ... ?

Statement (1) tells us that p is divisible by 14, which is 2×7. Therefore, we know that p has at least a 2 and a 7 in its prime box. However, we do not know anything else about the possible prime factors in p, so we cannot determine whether p is divisible by 168. For example, p could equal $2 \times 2 \times 2 \times 3 \times 7 = 168$, in which case the answer to the question is "yes, p is divisible by 168." Alternatively, p could equal $2 \times 7 = 14$, in which case the answer to the question is "no, p is NOT divisible by 168." Therefore Statement (1) is insufficient.

Statement (2): *p*
2, 2, 3, ... ?

Statement (2) tells us that p is divisible by 12, which is $2 \times 2 \times 3$. Therefore, we know that p has at least two 2's and a 3 in its prime box. However, we do not know anything else about p, so we cannot determine whether p is divisible by 168. For example, p could equal 168, in which case the answer to the question is "yes, p is divisible by 168." Alternatively, p could equal 12, in which case the answer to the question is "no, p is NOT divisible by 168." Statement (2) is insufficient.

INCORRECT:
Stmt's (1) & (2): *p*

~~2, 2, 2,~~
~~3, 7, ... ?~~

What about combining the information from Statements (1) and (2)? Can we simply take all of the primes from the two prime boxes we created, put them into a new prime box, and determine whether *p* is divisible by 168? Combining the primes from Statements (1) and (2), we *seem* to have three 2's, a 3, and a 7. That should be sufficient to prove that *p* is divisible by 168.

The short answer is no, we cannot do this. Consider the number 84. 84 is divisible by 14. It is also divisible by 12. Therefore, following from Statements (1) and (2), *p* could be 84. However, 84 is not divisible by 168. $84 = 2 \times 2 \times 3 \times 7$, so we are missing a needed 2.

BEWARE of prime numbers that appear in multiple prime boxes for the same variable!

CORRECT:
Stmt's (1) & (2): *p*

2, 2, 3, 7, ... ?

BOTH statements mention that *p* contains at least one 2 in its prime factorization. It is possible that these statements are referring to the SAME 2. Therefore, one of the 2's in Statement (2) OVERLAPS with the 2 from Statement (1). You have been given REDUNDANT information. The two boxes you made for Statements (1) and (2) are not truly different boxes. Rather, they are two different views of the same box (the prime box of *p*).

Thus, we have to eliminate the redundant 2 when we combine the two views of *p*'s prime box from Statements (1) and (2). Given both statements, we only know that *p* has two 2's, a 3, and a 7 in its prime box. INSUFFICIENT. The correct answer is (E): Statements (1) and (2) TOGETHER are NOT sufficient.

Note that if the problem were changed slightly, the answer would be different.

Is *pq* divisible by 168?

(1) *p* is divisible by 14
(2) *q* is divisible by 12

In this altered problem, the information in the two statements is NOT redundant. There is no overlap between the prime boxes, because the prime boxes belong to different variables (*p* and *q*). Statement (1) tells us that *p* has at least one 2 and one 7 in its prime box. Statement (2) tells us that *q* has at least two 2's and one 3 in its prime box. When we combine the two statements, we combine the prime boxes without removing any overlap, because there is no such overlap. As a result, we know that the product *pq* contains at least THREE 2's, one 3, and one 7 in its combined prime box. We can now answer the question "Is *pq* divisible by 168?" with a definitive "Yes," since the question is really asking whether *pq* contains at least three 2's, one 3, and one 7 in its prime box.

The correct answer to this problem is (C): Statements (1) and (2) TOGETHER are SUFFICIENT.

Greatest Common Factor and Least Common Multiple

Frequently on the GMAT, you may have to find the Greatest Common Factor (GCF) or Least Common Multiple (LCM) of a set of two or more numbers.

Greatest Common Factor (GCF): the largest divisor of two integers.
Least Common Multiple (LCM): the smallest multiple of two integers.

It is likely that you already know how to find both the GCF and the LCM. For example, when you reduce the fraction $\frac{9}{12}$ to $\frac{3}{4}$, you are dividing both the numerator (9) and denominator (12) by 3, which is the GCF of 9 and 12. When you add together the fractions $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}$, you convert the fractions to thirtieths: $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}=\frac{15}{30}+\frac{10}{30}+\frac{6}{30}=\frac{31}{30}$. Why thirtieths? The reason is that 30 is the LCM of the denominators: 2, 3, and 5.

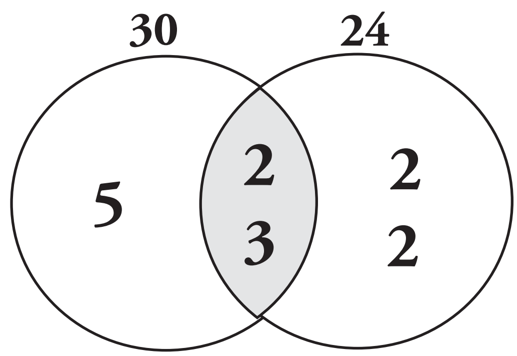
The GCF and the LCM can best be understood visually by using a Venn diagram.

Though finding the GCF and LCM of two small numbers is generally simple, for more difficult problems there are several approaches.

FINDING GCF AND LCM USING VENN DIAGRAMS

One way that you can visualize the GCF and LCM of two numbers is by placing prime factors into a **Venn diagram**—a diagram of circles showing the overlapping and non-overlapping elements of two sets. To find the GCF and LCM of two numbers using a Venn diagram, perform the following steps:

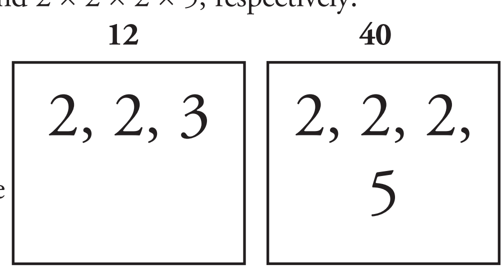
(1) Factor the numbers into primes.
(2) Create a Venn diagram.
(3) Place each common factor, including copies of common factors appearing more than once, into the shared area of the diagram (the shaded region to the right).
(4) Place the remaining (non-common) factors into the non-shared areas.



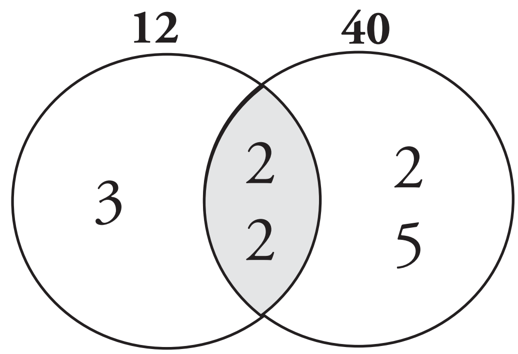
The Venn diagram above illustrates how to determine the GCF and LCM of 30 and 24. **The GCF is the product of primes in the overlapping region**: $2 \times 3 = 6$. **The LCM is the product of ALL primes in the diagram**: $5 \times 2 \times 3 \times 2 \times 2 = 120$.

Compute the GCF and LCM of 12 and 40 using the Venn diagram approach.

The prime factorizations of 12 and 40 are $2 \times 2 \times 3$ and $2 \times 2 \times 2 \times 5$, respectively:



The only common factors of 12 and 40 are two 2's. Therefore, we place two 2's in the shared area of the Venn diagram and remove them from BOTH prime factorizations. Then, place the remaining factors in the zones belonging exclusively to 12 and 40. These two outer regions **must have *no* primes in common**!



The GCF of 12 and 40 is therefore $2 \times 2 = 4$, the product of the primes in the **shared area**. (An easy way to remember this is that the "common factors" are in the "common area.")

The LCM is $2 \times 2 \times 2 \times 3 \times 5 = 120$, the product of **all** the primes in the diagram.

Note that if two numbers have NO primes in common, then their GCF is 1 and their LCM is simply their product. For example, 35 ($= 5 \times 7$) and 6 ($= 2 \times 3$) have no prime numbers in common, so their GCF is 1 and their LCM is $35 \times 6 = 210$.

Use prime columns to calculate the GCF and LCM of 2 or more numbers quickly.

FINDING GCF AND LCM USING PRIME COLUMNS

While Venn diagrams are helpful for visualizing the steps needed to compute the GCF and LCM, they can be cumbersome if you want to find the GCF or LCM of large numbers or of 3 or more numbers.

Prime columns is a technique that makes this process faster and easier. *Note: this discussion introduces exponents. If you are unfamiliar with exponents, please turn to Chapter 5.* Here are the steps:

(1) Calculate the prime factors of each integer.
(2) Create a column for each prime factor found within any of the integers.
(3) Create a row for each integer.
(4) In each cell of the table, place the prime factor raised to a *power*. This power counts how many copies of the column's prime factor appear in the prime box of the row's integer.

To calculate the GCF, take the LOWEST count of each prime factor found across ALL the integers. This counts the **shared primes**. To calculate the LCM, take the HIGHEST count of each prime factor found across ALL the integers. This counts **all the primes less the shared primes**.

Here is an example to demonstrate the method:

Find the GCF and LCM of 100, 140, and 250.

First, we need to find the prime factorizations of these numbers. $100 = 2 \times 2 \times 5 \times 5 = 2^2 \times 5^2$. $140 = 2 \times 2 \times 5 \times 7 = 2^2 \times 5 \times 7$. Finally, $250 = 2 \times 5 \times 5 \times 5 = 2 \times 5^3$.

Now, set up a table listing the prime factors of each of these integers in exponential notation. The different prime factors are 2, 5, and 7, so we need 3 columns.

Number:	**2**		**5**		**7**
100	2^2	$\times$	5^2	$\times$	–
140	2^2	$\times$	5^1	$\times$	7^1
250	2^1	$\times$	5^3		–

To calculate the GCF, we take the SMALLEST count (the lowest power) in any column. The reason is that the GCF is formed only out of the SHARED primes (in the overlapping part of the Venn diagram). The smallest count of the factor 2 is **one**, in 250 (= $2^1 \times 5^3$). The smallest count of the factor 5 is **one**, in 140 (= $2^2 \times 5^1 \times 7^1$). The smallest count of the factor 7 is **zero**, since 7 does not appear in 100 or in 250. Therefore the GCF is $2^1 \times 5^1 = 10$.

To calculate the LCM, we take the LARGEST count (the highest power) in any column. The reason is that the LCM is formed out of ALL the primes less the shared primes. The largest count of the factor 2 is **two**, in 140 (= $2^2 \times 5^1 \times 7^1$) and 100 (= $2^2 \times 5^2$). The largest count of the factor 5 is **three**, in 250 (= $2^1 \times 5^3$). The largest count of the factor 7 is **one**, in 140 (= $2^2 \times 5^1 \times 7^1$). Therefore the GCF is $2^2 \times 5^3 \times 7^1 = 3{,}500$.

Here is a tip for double-checking your work: for any integers a and b, the GCF times the LCM must equal $a \times b$!

Number:	**2**		**5**		**7**		
100	2^2	×	5^2		–		
140	2^2	×	5^1	×	7^1		
250	2^1	×	5^3		–		
GCF:	$\mathbf{2^1}$	×	$\mathbf{5^1}$			=	$\mathbf{2^1 \times 5^1 = 10}$
LCM:	$\mathbf{2^2}$	×	$\mathbf{5^3}$	×	$\mathbf{7^1}$	=	$\mathbf{2^2 \times 5^3 \times 7^1 = 3{,}500}$

FINDING GCF AND LCM USING PRIME BOXES OR FACTORIZATIONS

Finally, you can use a shortcut directly from the prime boxes or the prime factorizations to find the GCF and LCM. Once you get familiar with the Prime Columns method, you will see that you can just scan the boxes or the factorizations and take all the lowest powers to find the GCF and the highest powers to find the LCM.

What are the GCF and LCM of 30 and 24?

30	**24**
2, 3, 5	2, 2, 2, 3

The prime factorization of 30 is $2 \times 3 \times 5$.
The prime factorization of 24 is $2 \times 2 \times 2 \times 3$, or $2^3 \times 3$.
The GCF is $2 \times 3 = 6$.
The LCM is $2^3 \times 3 \times 5 = 120$.

Three properties of the GCF and LCM are worth noting:

(1) **(GCF of *m* and *n*) × (LCM of *m* and *n*) = *m* × *n*.** The reason for this is that the GCF is composed of the SHARED prime factors of *m* and *n*. The LCM is composed of all of the other, or NON-SHARED, prime factors of *m* and *n*.

(2) **The GCF of *m* and *n* cannot be larger than the difference between *m* and *n*.** For example, assume the GCF of *m* and *n* is 12. Thus, *m* and *n* are both multiples of 12. Consecutive multiples of 12 are 12 units apart on the number line. Therefore, *m* and *n* CANNOT be less than 12 units apart, or else they would not both be multiples of 12.

(3) **Consecutive multiples of *n* have a GCF of *n*.** For example, 8 and 12 are consecutive multiples of 4. Thus 4 is a common factor of both numbers. But 8 and 12 are exactly 4 units apart. Thus 4 is the greatest possible common factor of 8 and 12. (For this reason, the GCF of any two consecutive integers is 1, because both integers are multiples of 1 and the numbers are 1 unit apart.)

Other Applications of Primes & Divisibility

COUNTING FACTORS AND PRIMES

The GMAT can ask you to count factors of some number in several different ways. For example, consider the number 1,400. The prime factorization of this number is $2 \times 2 \times 2 \times 5 \times 5 \times 7$, or $2^3 \times 5^2 \times 7$ in **exponential notation**. Here are three different questions that the GMAT could ask you about this integer:

The GMAT can ask you to calculate the total number of factors, the total number of prime factors (length), or the total number of DIFFERENT prime factors of any integer.

Factors of 1,400 ($2^3 \times 5^2 \times 7$):

(1) How many different prime factors?

- May be phrased as "different prime factors" or "unique prime factors"
- Count each repeated prime factor only ONCE
- In this example, 2, 5, and 7 are distinct, so there are 3 different prime factors

(2) How many total prime factors (length)?

- Length is defined as the number of primes (not necessarily distinct) whose product is x (in this case, whose product is 1,400)
- Add the exponents of the prime factors. If there is no exponent, count it as 1.
- In this example, the length is $3 + 2 + 1 = 6$.

(3) How many total factors?

- Includes all factors, not necessarily just prime factors
- Can be determined using "factor pairs" approach, but this is cumbersome for larger numbers
- Advanced technique discussed later in this chapter
- Do not forget to include 1 as a factor!

Consider the number 252.
(a) How many unique prime factors of 252 are there?
(b) What is the length of 252 (as defined above)?
(c) How many total factors of 252 are there?

For (a), we can determine the number of unique prime factors by looking at the prime factorization of 252: $2 \times 2 \times 3 \times 3 \times 7$. There are 3 different prime factors in 252: 2, 3, and 7. We do NOT count repeated primes to answer this particular question.

For (b), the "length" of an integer is defined as the total number of primes that, when multiplied together, equal that integer. (Note: on the GMAT, any question that asks about the length of an integer will provide this definition of length, so you do not need to memorize it.) Again we can determine the TOTAL number of prime factors by looking at the prime factorization of 252: $2 \times 2 \times 3 \times 3 \times 7$. There are 5 total prime factors in 252: 2, 2, 3, 3, and 7. In other words, the length of an integer is just the total number of primes in the prime box of that integer. We DO count repeated primes to answer this particular question.

You can also answer this question by looking at the prime factorization in exponential form: $252 = 2^2 \times 3^2 \times 7$. Simply add the exponents: $2 + 2 + 1 = 5$. Notice that a number written in this form without an exponent has an **implicit exponent** of 1.

For (c), one way to determine the total number of factors is to determine the factor pairs of 252, using the process described earlier in this chapter. Simply start at 1 and "walk up" through all the integers, determining whether each is a factor. Meanwhile, the factors in the large column will naturally get smaller. You can stop once the small column "meets" the large column. For example, since the last entry in the large column is 18, you can stop searching once you have evaluated 17 as a possible factor. Once you have finished, you will notice there are 18 total factors of 252.

Small	Large
1	252
2	126
3	84
4	63
6	42
7	36
9	28
12	21
14	18

This method will be too cumbersome for larger numbers, so a more advanced method is introduced later in this chapter.

Perfect squares always have an odd number of factors; other integers always have an even number of factors.

PERFECT SQUARES, CUBES, ETC.

The GMAT occasionally tests properties of perfect squares, which are squares of other integers. The numbers 4 ($= 2^2$) and 25 ($= 5^2$) are examples of perfect squares. One special property of perfect squares is that **all perfect squares have an odd number of total factors**. Similarly, any integer that has an odd number of total factors MUST be a perfect square. All other non-square integers have an even number of factors. Why is this the case?

Think back to the factor pair exercises we have done so far. Factors come in pairs. If x and y are integers and $x \cdot y = z$, then x and y are a factor pair of z. However, if z is a perfect square, then in *one* of its factor pairs, x equals y. That is, in this particular pair we have $x \cdot x = z$, or $x^2 = z$. This means that we do not have TWO different numbers in this factor "pair." Rather, we have a single unpaired factor: the square root.

Consider the perfect square 36. It has 5 factor pairs that yield 36, as shown to the right. Notice that the FINAL pair is 6 and 6, so instead of $5 \times 2 = 10$ total factors, there are only 9 different factors of 36.

Small	Large
1	36
2	18
3	12
4	9
6	**6**

Notice also that any number that is not a perfect square will NEVER have an odd number of factors. That is because the only way to arrive at an odd number of factors is to have a factor pair in which the two factors are equal.

For larger numbers, it would be much more difficult to use the factor pair technique to prove that a number is a perfect square or that it has an odd number of factors. Thankfully, we can use a different approach. Notice that perfect squares are formed from the product of two copies of the same prime factors. For instance, $90^2 = (2 \times 3^2 \times 5)(2 \times 3^2 \times 5) = 2^2 \times 3^4 \times 5^2$. Therefore, **the prime factorization of a perfect square contains only even powers of primes**. It is also true that any number whose prime factorization contains only even powers of primes must be a perfect square.

Here are some examples.

$144 = 2^4 \times 3^2$ $\quad$ $9 = 3^2$

$36 = 2^2 \times 3^2$ $\quad$ $40{,}000 = 2^6 \times 5^4$

All of these integers are perfect squares.

By contrast, if a number's prime factorization contains any odd powers of primes, then the number is not a perfect square. For instance, $132{,}300 = 2^2 \times 3^3 \times 5^2 \times 7^2$ is not a perfect square, because the 3 is raised to an odd power. If this number is multiplied by 3, then the result, 396,900, is a perfect square: $396{,}900 = 2^2 \times 3^4 \times 5^2 \times 7^2$.

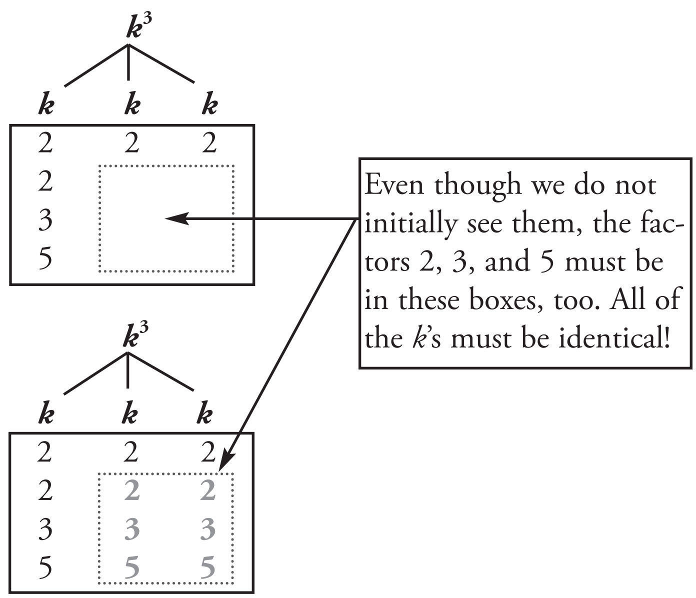
Prime factors of perfect squares MUST come in pairs; likewise, prime factors of perfect cubes MUST come in groups of 3!

The same logic used for perfect squares extends to perfect cubes and to other "perfect" powers. If a number is a perfect cube, then it is formed from three identical sets of primes, so all the powers of primes are multiples of 3 in the factorization of a perfect cube. For instance, $90^3 = (2 \times 3^2 \times 5)\ (2 \times 3^2 \times 5)\ (2 \times 3^2 \times 5) = 2^3 \times 3^6 \times 5^3$.

If k^3 is divisible by 240, what is the least possible value of integer k?
(A) 12 (B) 30 (C) 60 (D) 90 (E) 120

The prime box of k^3 contains the prime factors of 240, which are 2, 2, 2, 2, 3, and 5. We know that the prime factors of k^3 should be the prime factors of k appearing in sets of three, so we should distribute the prime factors of k^3 into three columns to represent the prime factors of k, as shown below.

We see a complete set of three 2's in the prime box of k^3, so k must have a factor of 2. However, there is a fourth 2 left over. That additional factor of 2 must be from k as well, so we assign it to one of the component k columns. We have an incomplete set of 3's in the prime box of k^3, but we can still infer that k has a factor of 3; otherwise k^3 would not have any. Similarly, k^3 has a single 5 in its prime box, but that factor must be one of the factors of k as well. Thus, k has 2, 2, 3, and 5 in its prime box, so k must be a multiple of 60.



The correct answer is **C**.

FACTORIALS AND DIVISIBILITY

The factorial of N, symbolized by $N!$, is the product of all integers from 1 up to and including N. For instance, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Because it is the product of all the integers from 1 to N, any factorial $N!$ must be divisible by all integers from 1 to N. This follows directly from the Factor Foundation Rule. Another way of saying this is that ***N*! is a multiple of all the integers from 1 to *N***.

This fact works in concert with other properties of divisibility and multiples. For instance, the quantity 10! + 7 must be a multiple of 7, because both 10! and 7 are multiples of 7. 10! + 15 must be a multiple of 15, because 10! is divisible by 5 and 3, and 15 is divisible by 5 and 3. Thus, both numbers are divisible by 15, and the sum is divisible by 15. Finally, 10! + 11! is a multiple of any integer from 1 to 10, because every integer between 1 and 10 inclusive is a factor of both 10! and 11!, separately.

$N!$ is the product of all positive integers smaller than or equal to N. Therefore, $N!$ MUST be divisible by all integers from 1 to N.

Because the factorial $N!$ contains all the integers from 1 up to the number N, it follows that any smaller factorial divides evenly into any larger factorial. For example, 9! is divisible by 8! or by the factorial of any smaller positive integer. In a quotient of two factorials, the smaller factorial cancels completely. For instance, consider $\frac{8!}{5!}$:

$$\frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 8 \cdot 7 \cdot 6 = 336.$$

Remainder Theory (Advanced)

The remainder is an important concept on the GMAT that is tied very closely with divisibility. A remainder is defined as the integer portion of the **dividend** (or numerator) that is not evenly divisible by the **divisor** (or denominator). For example, 23 is not evenly divisible by 4. When you divide 23 by 4, you get a **remainder** of 3 that cannot be divided out, because $23 = 5 \times 4 + 3$. Here is this example written in fractional notation:

Dividend **Remainder**

$$\frac{23}{4} = 5 + \frac{3}{4}$$

Divisor

Quotient

In a remainder problem, the dividend, divisor, quotient, and remainder must ALL be integers.

The **quotient** is the resulting integer portion that CAN be divided out (in this case, the quotient is 5). Note that the dividend, divisor, quotient and remainder will ALWAYS be integers. Sometimes, the quotient may be zero! For instance, when 3 is divided by 5, the remainder is 3 (because 0 is the biggest multiple of 5 that can be divided out of 3).

Algebraically, this relationship can be written as:

Dividend **Remainder**

$$\frac{x}{N} = Q + \frac{R}{N}$$

Divisor

Quotient

This framework is often easiest to use on GMAT problems when you multiply through by the divisor N:

Dividend **Remainder**

$$x = Q \cdot N + R$$

Quotient **Divisor**

(Example: 23 = 5 × 4 + 3)

Again, remember that x, Q, N, and R all must ALL be integers.

<u>CREATING NUMBERS WITH A CERTAIN REMAINDER</u>

Rather than requiring you to find remainders, some GMAT problems require you to generate arbitrary numbers that yield a certain remainder upon division. If you need a number that leaves remainder R upon division by N, simply take any multiple of N and add R to it. This is equivalent to writing $x = Q \times N + R$, since $Q \times N$ is a multiple of N.

For example, if you need a number that leaves a remainder of 5 when divided by 7, you can pick 14 (a multiple of 7) and add 5 to get 19. You could also pick 7 (a multiple of 7) and add 5 to get 12. As another example, if you need a number that leaves a remainder of 3 when divided by 4, you can pick 20 (a multiple of 4) and add 3 to get 23.

Consider this example:

> If x has a remainder of 3 when divided by 7 and y has a remainder of 2 when divided by 7, what is the remainder of $x + y$ when divided by 7?

One way to solve this problem is simply to pick numbers for x and y using the method described above. For example, $14 + 3 = 17$ could be x, and $7 + 2 = 9$ could be y. Adding them together, we see that $17 + 9 = 26$, which has a remainder of 5 when divided by 7. Notice that this remainder is the sum of the remainders of x and y.

RANGE OF POSSIBLE REMAINDERS

When you divide an integer by 7, the remainder could be 0, 1, 2, 3, 4, 5, or 6. Notice that you cannot have a negative remainder or a remainder larger than 7, and that you have exactly 7 possible remainders. This pattern can be generalized. When you divide an integer by a positive integer N, the possible remainders range from 0 to $(N - 1)$. There are thus N possible remainders. Negative remainders are not possible, nor are remainders larger than N.

The remainder of any number MUST be non-negative and smaller than the divisor.

> If $a \div b$ yields a remainder of 5, $c \div d$ yields a remainder of 8, and a, b, c and d are all integers, what is the smallest possible value for $b + d$?

Since the remainder must be smaller than the divisor, 5 must be smaller than b. b must be an integer, so b is at least 6. Similarly, 8 must be smaller than d, and d must be an integer, so d must be at least 9. Therefore the smallest possible value for $b + d$ is $6 + 9 = 15$.

REMAINDER OF 0

If x divided by y yields a remainder of 0 (commonly referred to as "no remainder"), then x is divisible by y. Conversely, if x is divisible by y, then x divided by y yields a remainder of 0 (or "no remainder").

Similarly, if x divided by y yields a remainder greater than 0, then x is NOT divisible by y, and vice versa.

ARITHMETIC WITH REMAINDERS

Here are two useful facts about arithmetic with remainders, when you have the same divisor throughout.

(1) You can add and subtract remainders directly, as long as you correct excess or negative remainders. "Excess remainders" are remainders larger than or equal to the divisor. To correct excess or negative remainders, add or subtract the divisor.

For instance, if x leaves a remainder of 4 after division by 7 and y leaves a remainder of 2 after division by 7, then $x + y$ leaves a remainder of $4 + 2 = 6$ after division by 7. You do not need to pick numbers for x and y to calculate this result.

If x leaves a remainder of 4 after division by 7 and z leaves a remainder of 5 after division by 7, then adding the remainders together yields 9. This number is too high, however. The remainder must be non-negative and less than 7. Notice that we can take an additional 7 out of the remainder, because 7 is the **excess** portion of our remainder. Thus $x + z$ leaves a remainder of $9 - 7 = 2$ after division by 7.

With the same x and z, subtraction of the remainders gives -1, which is also an unacceptable remainder (it must be non-negative). In this case, add an extra 7 to see that $x - z$ leaves a remainder of 6 after division by 7.

You can generally pick numbers to prove that this process works. Let us pick $x = 25$ and $z = 12$:

$$\underbrace{25+12=37=5}_{\text{Quotient}}\cdot\underbrace{7}_{\text{Divisor}}+\underbrace{2}_{\text{Remainder}}$$

$$\underbrace{25-12=13=1}_{\text{Quotient}}\cdot\underbrace{7}_{\text{Divisor}}+\underbrace{6}_{\text{Remainder}}$$

(2) You can multiply remainders, as long as you correct excess remainders at the end.

Most solutions to remainder problems can be double-checked by picking numbers.

Again, if x has a remainder of 4 upon division by 7 and z has a remainder of 5 upon division by 7, then 4×5 gives 20. Two additional 7's can be taken out of this remainder, so $x \cdot z$ will have remainder 6 upon division by 7. We can prove this by again picking $x = 25$ and $z = 12$:

$$\underbrace{25\times12=300=42}_{\text{Quotient}}\cdot\underbrace{7}_{\text{Divisor}}+\underbrace{6}_{\text{Remainder}}$$

Advanced GCF and LCM Techniques (Advanced)

Previously in this chapter, we discussed how to calculate the GCM and LCF of two or more numbers. A more difficult task is to determine what combinations of numbers could lead to a specific GCF or LCM.

Is the integer z divisible by 6?

(1) The greatest common factor of z and 12 is 3.
(2) The greatest common factor of z and 15 is 15.

Think back to how we calculated the GCF for a set of numbers. We determined the prime factors of each number and then took each prime factor to the LOWEST power it appeared in any of the factorizations. In this problem, we are TOLD what the GCF is. We can use the prime columns method to determine what conclusions can be drawn from each of these statements.

Statement (1) tells us that z and 12 ($2 \times 2 \times 3$) have a GCF of 3. Set that information up in a prime columns table to figure out what we can deduce about the prime factors of z.

Notice that the GCF of 12 and z contains a 3. Since the GCF contains each prime factor to the power it appears the LEAST, we know that z must also contain at least one 3. Therefore, z is divisible by 3.

Number:	2		3
z	?	×	?
12	2^2	×	3^1
GCF:	–	×	$\mathbf{3^1}$

Notice also that the GCF contains NO 2's. Since 12 contains two 2's, *z* must not contain any 2's. Therefore, *z* is NOT divisible by 2. Since *z* is not divisible by 2, it cannot be divisible by 6. SUFFICIENT.

Statement (2) tells us that *z* and 15 (3×5) have a GCF of 15. We can set that up in a prime columns table to figure out what we can deduce about the prime factors of *z*:

The GCF of 15 and *z* contains a 3. Since the GCF contains each prime factor to the power it appears the LEAST, we know that *z* must also contain at least one 3. Therefore, *z* is divisible by 3.

Number:	**3**		**5**
z	?	×	?
15	3^1	×	5^1
GCF:	$\mathbf{3^1}$	×	$\mathbf{5^1}$

Some difficult GCF/LCM problems require that you use the GCF/LCM to try to deduce what the original numbers could have been.

Also, the GCF contains a 5. Since the GCF contains each prime factor to the power it appears the LEAST, we know that *z* must also contain a 5. Therefore, *z* is divisible by 5.

However, this does NOT tells us whether *z* contains any 2's. We need *z* to contain at least one 2 and at least one 3 in its prime factorization for it to be divisible by 6. If *z* had 2 as a prime factor, 2 would still not be part of the GCF, because 15 has no 2's. Thus we cannot tell whether *z* has a 2 in its prime factorization. INSUFFICIENT. The correct answer is (A): Statement (1) ALONE is sufficient, but statement (2) alone is not sufficient.

If the LCM of *a* and 12 is 36, what are the possible values of *a*?

As in the example problem above, we can use the prime columns technique to draw conclusions about the prime factors of *a*.

First, notice that *a* cannot be larger than 36. The LCM of two or more integers is always AT LEAST as large as any of the integers. Therefore the maximum value of *a* is 36.

Number:	**2**		**3**
a	?	×	?
12	2^2	×	3^1
LCM:	$\mathbf{2^2}$	×	$\mathbf{3^2}$

Next, notice that the LCM of 12 and *a* contains two 2's. Since the LCM contains each prime factor to the power it appears the MOST, we know that *a* cannot contain more than two 2's. It does not necessarily to contain any twos, so *a* can contain zero, one or two 2's.

Finally, observe that the LCM of 12 and *a* contains two 3's. But 12 only contains ONE 3. The 3^2 factor in the LCM must have come from the prime factorization of *a*. Thus we know that *a* contains exactly two 3's.

Since *a* must contain exactly two 3's, and can contain no 2's, one 2, or two 2's, *a* could be $3 \times 3 = 9$, $3 \times 3 \times 2 = 18$, or $3 \times 3 \times 2 \times 2 = 36$. Thus 9, 18, and 36 are the possible values of *a*.

Counting Total Factors (Advanced)

We have discussed using the **factor pair** method to determine the number of total factors of an integer. The problem with this method is that it is slow, tedious, and prone to error. These problems are compounded when the number being analyzed has a large number of factors. Therefore, we need a general method to apply to more difficult problems of this type.

How many different factors does 2,000 have?

It would take a very long time to list all of the factors of 2,000. However, prime factorization can shorten the process considerably. First, factor 2,000 into primes: $2{,}000 = 2^4 \times 5^3$. The key to this method is to consider each distinct prime factor separately.

Some basic combinatorics theory can help streamline the process of counting factors for any integer.

Consider the prime factor 2 first. Because the prime factorization of 2,000 contains four 2's, there are **five possibilities for the number of 2's** in any factor of 2,000: none, one, two, three, or four. (Do not forget the possibility of NO occurrences! For example, 5 is a factor of 2,000, and 5 does not have ANY 2's in its prime box.)

Next, consider the prime factor 5. There are three 5's, so there are **four possibilities for the number of 5's** in a factor of 2,000: none, one, two, or three. (Again, do not forget the possibility of NO occurrences of 5.) Any number with more than four 2's in its prime box cannot be a factor of 2,000.

In general, if a prime factor appears to the *N*th power, then there are $(N + 1)$ possibilities for the occurrences of that prime factor. This is true for each of the individual prime factors of any number.

We can borrow a principle from the field of combinatorics called the **Fundamental Counting Principle** to simplify the calculation of the number of prime factors in 2,000. The Fundamental Counting Principle states that if you must make a number of separate decisions, then MULTIPLY the numbers of ways to make each *individual* decision to find the number of ways to make *all* the decisions. (For an elaboration of this principle, see the "Combinatorics" chapter of Manhattan GMAT's *Word Translations* Strategy Guide.)

The number of 2's and the number of 5's to include in a factor of 2,000 are two individual decisions we must make. These two choices are INDEPENDENT of one another, so the total number of factors of 2,000 must be $(4 + 1)(3 + 1) = 5 \times 4 = 20$ different factors.

The logic behind this process can also be represented in the following table of factors. (Note that there is no reason to make this table, unless you are interested in the specific factors themselves. It simply illustrates the reasoning behind multiplying the possibilities.)

	2^0	2^1	2^2	2^3	2^4
5^0	1	2	4	8	16
5^1	5	10	20	40	80
5^2	25	50	100	200	400
5^3	125	250	500	1,000	2,000

Notice that the factor in the top left corner has no 5's and no 2's in the factor. That factor is 1.

Although a table like the one above cannot be easily set up for more than two prime factors, the process can be generalized to numbers with more than two prime factors. Again, simply take the exponents on EACH prime factor in the factorization, add one to each of those exponents, and multiply the results together.

For instance, $9{,}450 = 2^1 \times 3^3 \times 5^2 \times 7^1$, so 9,450 has $(1 + 1)(3 + 1)(2 + 1)(1 + 1) = 48$ different factors.

Solve factor counting problems quickly by writing the prime factorization in exponential notation, adding 1 to all of the exponents, and multiplying.

Problem Set

For problems #1–11, use one or more prime boxes to answer each question: YES, NO, or CANNOT BE DETERMINED. If your answer is CANNOT BE DETERMINED, use two numerical examples to show how the problem could go either way. All variables in problems #1 through #11 are assumed to be integers unless otherwise indicated.

1. If a is divided by 7 or by 18, an integer results. Is $\frac{a}{42}$ an integer?

2. If 80 is a factor of r, is 15 a factor of r?

3. Given that 7 is a factor of n and 7 is a factor of p, is $n + p$ divisible by 7?

4. Given that 8 is not a factor of g, is 8 a factor of $2g$?

5. If j is divisible by 12 and 10, is j divisible by 24?

6. If 12 is a factor of xyz, is 12 a factor of xy?

7. Given that 6 is a divisor of r and r is a factor of s, is 6 a factor of s?

8. If 24 is a factor of h and 28 is a factor of k, must 21 be a factor of hk?

9. If 6 is not a factor of d, is $12d$ divisible by 6?

10. If k is divisible by 6 and $3k$ is not divisible by 5, is k divisible by 10?

11. If 60 is a factor of u, is 18 a factor of u?

Solve Problems #12–20:

12. What is the greatest common factor of 420 and 660?

13. What is the least common multiple of 18 and 24?

14. If w is a prime number greater than 3, and $z = 36w$, what is the least common multiple of z and $6w$, in terms of w?

15. If $y = 30p$, and p is prime, what is the greatest common factor of y and $14p$, in terms of p?

16. A skeet shooting competition awards prizes as follows: the first place winner receives 11 points, the second place winner receives 7 points, the third place finisher receives 5 points, and the fourth place finisher receives 2 points. No other prizes are awarded. John competes in the skeet shooting competition several times and receives points every time he competes. If the product of all of the points he receives equals 84,700, how many times does he participate in the competition?

1. **YES:**

a

2, 3, 3, 7, ... ?

If a is divisible by 7 and by 18, its prime factors include 2, 3, 3, and 7, as indicated by the prime box to the left. Therefore, any integer that can be constructed as a product of any of these prime factors is also a factor of a. $42 = 2 \times 3 \times 7$. Therefore, 42 is also a factor of a.

2. **CANNOT BE DETERMINED:**

r

2, 2, 2, 2, 5, ... ?

If r is divisible by 80, its prime factors include 2, 2, 2, 2, and 5, as indicated by the prime box to the left. Therefore, any integer that can be constructed as a product of any of these prime factors is also a factor of r. $15 = 3 \times 5$. Since the prime factor 3 is not in the prime box, we cannot determine whether 15 is a factor of r. Remember, this prime box represents a partial listing of the prime factors of r. There could be additional prime factors.

3. **YES:** If 2 numbers are both multiples of the same number, then their SUM is also a multiple of that same number. Since n and p share the common factor 7, the sum of n and p must also be divisible by 7.

4. **CANNOT BE DETERMINED:**

2*g*

2, *g*

(not 2, 2, 2)

In order for 8 to be a factor of $2g$, we would need two more 2's in the prime box. By the Factor Foundation Rule, g would need to be divisible by 4. We know that g is not divisible by 8, but there are certainly integers that are divisible by 4 and not by 8, such as 4, 12, 20, 28, etc. However, while we cannot conclude that g is **not** divisible by 4, we cannot be certain that g **is** divisible by 4, either.

5. **CANNOT BE DETERMINED:**

12	**10**		***j***
2, 2, 3, ... ?	, 2, 5, ... ?	→	2, 2, 3, 5, ... ?

If j is divisible by 12 and by 10, its prime factors include 2, 2, 3, and 5, as indicated by the prime box to the left. There are only TWO 2's that are definitely in the prime factorization of j, because the 2 in the prime factorization of 10 may be REDUNDANT—that is, it may be the SAME 2 as one of the 2's in the prime factorization of 12.

$24 = 2 \times 2 \times 2 \times 3$. There are only two 2's in the prime box of j; 24 requires three 2's. Therefore, 24 is not necessarily a factor of j.

As another way to prove that we cannot determine whether 24 is a factor of j, consider 60. The number 60 is divisible by both 12 and 10. However, it is NOT divisible by 24. Therefore, j could equal 60, in which case it is not divisible by 24. Alternatively, j could equal 120, in which case it IS divisible by 24.

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